Self-consistent beam distributions with space charge and dispersion in a circular ring lattice

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(Received 30 October 1997)

The interplay between dispersion and space charge in circular accelerators or storage rings is investigated by looking for self-consistent, stationary solutions of the Vlasov-Poisson equation in the form of generalized Kapchinsky-Vladimirsky (KV) distributions. The smooth approximation is assumed. The results show a growth of the rms quantities describing the beam distribution with the longitudinal momentum spread, and the tune depression. This growth, however, is modest for realistic values of these parameters in strong focusing systems. [S1063-651X(98)06204-7]

PACS number(s): 29.20.Lq, 41.75.Fr, 41.85.Lc

I. INTRODUCTION

Very little can be found in the accelerator physics literature on the problem of the combined effect of space charge and dispersion [1,2]. An obvious reason is that so far the beam intensity in circular accelerators has never been high enough to justify a treatment of dispersion different from the one usually done in a single-particle perspective. Some advanced accelerator applications, however, require very high-current beams for which the space charge effects are expected to play an important role. Examples include heavy ion fusion drives, high energy boosters, accumulator rings, and spallation neutron sources.

In this context, the construction of a small electron ring to study a highly tune-depressed beam has been proposed at the University of Maryland [3]. The goal is to produce, maintain, and study a beam with a tune depression in the range 0.2–0.4. In this range of highly space charge dominated beams the answer to the question of whether the usual single-particle treatment of dispersion is still justified is not obvious. Also, it was important for the purposes of the ring design to understand the combined effects of space charge and dispersion in shaping the beam size. In order to get an insight into the scale of the problem, we studied self-consistent, stationary solutions of the proper Vlasov-Poisson equation including the dispersive term. Specifically, in this paper we explore a particular kind of distribution that generalizes the usual KV [4] beam to the case where dispersion is present.

A simplified model of the electron ring dynamics has been assumed, in which the external focusing functions and the radius of curvature are constant. Moreover all the nonlinearities due to the external focusing are neglected as well as all the chromatic terms.

The structure of the paper is as follows. After some general remarks on dispersion (Sec. II), we introduce the Hamiltonian for our model and the general form of the Vlasov-Poisson equation. Next, in Sec. III we write and solve the Poisson equation for the particular case of a KV beam where all the particles are equally off-momentum. In Sec. IV we write the Poisson equation for the case where the beam has a generalized KV distribution in the transverse plane and a Gaussian distribution for the longitudinal momentum spread. The following section contains a discussion of the numerical solution for the Poisson equation. Finally we treat the problem of the evolution of a beam through injection from a straight channel to a dispersive channel (Sec. VIII).

II. DISPERSION

Dispersion describes the deviation of a particle from the reference orbit due to a deviation from the design momentum or energy (see, e.g., [5,6]). The dispersion function $D(z)$ is defined as the solution of the equation:

$$D(z)''+k_x(z)D(z)=\frac{1}{\rho(z)}, \tag{1}$$

with a prime indicating a derivative with respect to z.

In a multiparticle perspective we are interested in describing a particle distribution function $f(x,p_x,y,p_y,\delta,z)$, and how the distribution itself or its moments are affected by the presence of dispersion. Consider for the moment a beam of noninteracting particles. From the Vlasov equation associated with the Hamiltonian (2),

$$H=\frac{1}{2}(p_x^2+p_y^2)+\frac{1}{2}[k_x(z)x^2+k_y(z)y^2]-\frac{\delta}{\rho(z)}x, \tag{2}$$

we can easily derive the differential equations for the second moments of the distribution (in this context we can ignore the motion in the vertical plane):

$$\frac{d}{dz}\langle x^2\rangle-2\langle xp_x\rangle=0, \tag{3a}$$

$$\frac{d}{dz}\langle p_x^2\rangle+2k_x(z)\langle xp_x\rangle=\frac{2}{\rho}\langle p_x\delta\rangle, \tag{3b}$$

$$\frac{d}{dz}\langle xp_x\rangle-\langle p_x^2\rangle+k_x(z)\langle x^2\rangle=\frac{1}{\rho}\langle x\delta\rangle, \tag{3c}$$

$$\frac{d}{dz}\langle x\delta\rangle=\langle p_x\delta\rangle, \tag{4a}$$

$$\frac{d}{dz}\langle p_x\delta\rangle=-k_x(z)\langle x\delta\rangle+\frac{1}{\rho}\langle\delta^2\rangle. \tag{4b}$$

The brackets denote the averaging over the phase space variables according to

$$\langle\xi_i\xi_j\rangle=\frac{\int\xi_i\xi_j f(x,p_x,y,p_y,\delta)d\mu}{\int f(x,p_x,y,p_y,\delta)d\mu},$$

where ξ_i, ξ_j are any canonical variables, and $d\mu=dxdp_xdydp_yd\delta$. The last two equations (4) can be combined into a single differential equation

$$\frac{d^2}{dz^2}\langle x\delta\rangle+k_x\langle x\delta\rangle=\frac{1}{\rho}\langle\delta^2\rangle. \tag{5}$$

From comparison between Eqs. (1) and (5) we conclude

$$\langle x\delta\rangle=\langle\delta^2\rangle D(z),$$
$$\langle p_x\delta\rangle=\langle\delta^2\rangle D(z)'. \tag{6}$$

After substituting Eqs. (6) into (3) we see that Eq. (3) turns into an inhomogeneous linear system, a particular solution of which, as it can be easily verified by direct substitution, is given by $\langle x^2\rangle=\langle\delta^2\rangle D(z)^2$, $\langle p_x^2\rangle=\langle\delta^2\rangle D'(z)^2$, $\langle xp_x\rangle=\langle\delta^2\rangle D(z)D'(z)$. Therefore, we can write the general solution as a superposition of that particular solution and the general solution of the homogeneous part of the differential system, which we indicate with the subscript ''o'':

$$\langle x^2\rangle=\langle x^2\rangle_o+\langle\delta^2\rangle D(z)^2,$$
$$\langle p_x^2\rangle=\langle p_x^2\rangle_o+\langle\delta^2\rangle D'(z)^2, \tag{7}$$
$$\langle xp_x\rangle=\langle xp_x\rangle_o+\langle\delta^2\rangle D(z)D'(z).$$

In the general case the solution of the homogeneous part of Eq. (3) will also depend on $\langle\delta^2\rangle$. In those cases [7] where the dependence is of an order higher than $\langle\delta^2\rangle$, we see from Eq. (7) that the dispersion function weights the dependence of the moments on $\langle\delta^2\rangle$. Moreover, the $\langle\delta^2\rangle$-independent part of the solution of the homogeneous part of Eq. (3) can be interpreted physically as describing the moments of the distribution either in the limit of vanishing momentum spread ($\langle\delta^2\rangle=0$), or at those locations in z where the dispersion function vanishes.

One of the questions we want to address in this paper is how Eqs. (7) changes when we allow space charge effects to enter the picture.

A consequence of the presence of space charge is to modify the strength of the effective focusing forces acting on the particles and therefore to *depress* the tune ν_{ox}. For a round KV beam [8] of radius a in the smooth approximation, the depressed tune ν_x reads

$$\nu_x^2=\nu_{ox}^2-\frac{K}{a^2}\rho_o^2, \tag{8}$$

where $K=2I(I_o\beta^3\gamma^3)$ is the generalized perveance [6] with I being the beam current, β, γ the relativistic factors, and $I_o\simeq 17$ kA for electrons. In the absence of space charge $D=\rho_o/\nu_{ox}^2$. We can question whether in the presence of space charge the expression for the second moment of the distribution can be recovered from Eqs. (7) by the change $\nu_x\leftrightarrow\nu_{ox}$ in the expression for the dispersion, where ν_{ox} is the undepressed tune and ν_x is the depressed tune for the beam distribution in the absence of dispersion [9].

As we will show, the estimate we get in this way, while working for a moderate space charge, fails for higher depressed tune, giving much larger values than the correct self-consistent theory. This is to be expected since the reshaping of the beam distribution due to dispersion carries a change in the tune.

In particular, since according to Eqs. (7) the presence of dispersion enlarges transversally the beam size, the space charge forces become weaker. As a consequence, the tune becomes less depressed than in the case with no dispersion.

III. THE VLASOV-POISSON EQUATIONS

Our model is described by a Hamiltonian [5] $H=H_\perp+H_\parallel$, where $H_\parallel=(m^2c^4/E_o^2)\delta^2$ is a purely longitudinal term and

$$H_\perp=\frac{1}{2}(p_x^2+p_y^2)+\frac{k}{2}(x^2+y^2)-\frac{x}{\rho_o}\delta+g_o\psi(x,y) \tag{9}$$

(with $g_o=q/mv_z^2\gamma^3$).

The Hamiltonian refers to a beam of particles of charge q and mass m in a smooth circular channel where both the external focusing function k and the radius of curvature ρ_o are z-independent. Also, we assume that the external focusing is the same in the horizontal and vertical plane. The self-force is described by the potential ψ. The design momentum, longitudinal velocity, and the corresponding relativistic factor are p_o, v_z, and γ.

In the model we neglect all the nonlinearities coming from the external focusing as well as all the chromatic terms since they are of third order. We also ignore space charge effects due to the finite curvature of the beam.

Moreover, since the Hamiltonian is time independent (no beam acceleration) the momentum deviation δ is a constant of the motion. Clearly $H_\perp$ is also an integral of the motion. By choosing a z-independent potential ψ describing the self-force, we neglect the effects of the longitudinal space charge. By doing so the model is understood to describe the dynamics of continuous (unbunched) beams with a thermal energy spread. Clearly the effects of the energy spread induced by the bunch edge effects are not captured in our model, and it would probably be rather challenging to incorporate them in a self-consistent treatment with dispersion. For a non-self-consistent description of these effects for a single pass through bending, see [10].

We want to search for self-consistent solutions $f(x,p_x,y,p_y,\delta)$ of the Vlasov-Poisson equation associated with H:

$$\frac{\partial f}{\partial z}+\{f,H\}=0,\quad \nabla^2\psi=-\frac{q}{\epsilon_0}n(x,y), \tag{10}$$

where $n(x,y)$ is the beam density

$$n(x,y)=\int\int\int f(x,p_x,y,p_y,\delta)d\delta dp_x dp_y. \tag{11}$$

In particular, we want to look for stationary solutions $\partial f/\partial z=0$. We recall that any function of the integrals of motion of a Hamiltonian system is a stationary solution of the corresponding Vlasov equation. Therefore, a particular stationary solution of the Vlasov equation associated with the Hamiltonian H is given by

$$f(x,p_x,y,p_y,\delta)=f_{\parallel}(\delta)f_{\perp}(H_{\perp}).$$

In this paper we consider a distribution in the transverse Hamiltonian $f_{\perp}(H_{\perp})$ in the form of a Dirac δ function. In the absence of dispersion such a choice leads to the usual KV beam. For the distribution in the longitudinal momentum spread we discuss two cases: a monochromatic distribution and a Gaussian distribution.

IV. MONOCHROMATIC KV BEAM

First of all, consider the particular choice

$$f_{\parallel}(\delta)=\hat{\delta}(\delta-\delta_o), \tag{12a}$$

$$f_{H_{\perp}}(H_{\perp})=f_o\hat{\delta}(H_{\perp}-H_o). \tag{12b}$$

Here $\hat{\delta}$ is the Dirac delta function and f_o is a constant. Such a distribution describes a beam of particles that are off-momentum by the same amount δ_o. Notice that for $\delta=\delta_o=0$ we recover the usual KV beam.

The space density associated with f is given by

$$\begin{aligned}n(x,y)&=f_o\int\int\int dp_x dp_y d\ \delta\hat{\delta}(\delta-\delta_o)\hat{\delta}(H_{\perp}-H_o)\\&=2\pi f_o\mathscr{H}\left(H_o-\frac{k}{2}(x^2+y^2)+\frac{\delta_o}{\rho_o}x-g_o\psi\right),\end{aligned}$$

where $\mathscr{H}$ is the Heaviside step function. The Poisson equation for the self-potential reads

$$\nabla^2\psi=-\frac{q}{\epsilon_o}2\pi f_o\mathscr{H}\left(H_o-\frac{k}{2}(x^2+y^2)+\frac{\delta_o}{\rho_o}x-g_o\psi\right). \tag{13}$$

One can verify that a solution is given by

$$\psi(x,y)=-\frac{q\pi f_o}{2\epsilon_0}\left[\left(x-\frac{\delta_o}{\rho_o k}\right)^2+y^2\right] \tag{14}$$

for $[x-\delta_o/(\rho_o k_x)]^2+y^2\leqslant a^2$, and

$$\psi(x,y)=-\frac{q\pi f_o a^2}{\epsilon_0}\left(\ln\left\{\frac{1}{a}\left[\left(x-\frac{\delta_o}{\rho_o k}\right)^2+y^2\right]^{1/2}\right\}+\frac{1}{2}\right) \tag{15}$$

for $[x-\delta_o/(\rho_o k_x)]^2+y^2>a^2$, where

$$a^2=\frac{1}{2}\left(k_x-g_o\frac{q\pi f_o}{\epsilon_0}\right)^{-1}\left(H_o+\frac{\delta_o^2}{2\rho_o^2 k_x}\right). \tag{16}$$

The calculation shows that in the presence of a momentum deviation represented by a δ function the beam density is that of an off-centered KV beam of radius a. The amount of the deviation from the axis is given by $\delta_o D$, with the dispersion function $D=1/(\rho_o k_x)$ given in terms of the undepressed focusing function. In other terms, particles of a monochromatic KV round beam, from the view point of dispersion, behave like single particles.

V. GENERALIZED KV BEAM

Next, we look for solutions of the Vlasov-Poisson equations describing a beam with a Gaussian-like distribution in the longitudinal momentum and a KV-beam-like distribution in the transverse plane:

$$f_{\parallel}(\delta)=\frac{1}{\delta_o\sqrt{\pi}}e^{-\delta^2/\delta_o^2}, \tag{17a}$$

$$f_{H_{\perp}}(H_{\perp})=f_o\hat{\delta}(H_{\perp}-H_o). \tag{17b}$$

We notice at this point that the resulting distribution will not have a perfect Gaussian character in the longitudinal momentum spread, because the term $f_{H_{\perp}}(H_{\perp})$ also depends on δ. However, in the range of parameters considered in this paper the deviation from a pure Gaussian distribution will be relatively small. In particular, the rms longitudinal momentum spread $\sigma_{\delta}=\sqrt{\langle\delta_o^2\rangle}$ will differ from $\delta_o/\sqrt{2}$ by a few percents.

Observe that in the limit $\delta_o\to 0$, we recover the usual KV distribution. In the following we will refer to the distribution described by Eq. (17) as a ''generalized KV beam.''

The corresponding space density is

$$n(x,y)=2\pi f_o\frac{1}{\delta_o\sqrt{\pi}}\int_{-\infty}^{\infty}\mathscr{H}\left(\lambda(x,y)+\frac{\delta_o x}{\rho_o}t\right)e^{-t^2}dt, \tag{18}$$

where we have defined

$$\lambda(x,y)=H_o-\frac{k_x}{2}x^2-\frac{k_y}{2}y^2-g_o\psi(x,y). \tag{19}$$

The integral in Eq. (18) can be easily carried out and expressed in terms of the error function,

$$n(x,y)=\pi f_o\left[\mathrm{erf}\left(\frac{\lambda(x,y)\rho_o}{\delta_o|x|}\right)+1\right], \tag{20}$$

with

$$\mathrm{erf}(\tau)=\frac{2}{\sqrt{\pi}}\int_0^{\tau}e^{-t^2}dt.$$

The Poisson equation then reads

$$\nabla^2\psi=-\frac{q}{\epsilon_o}\pi f_o\left\{\mathrm{erf}\left[\frac{\rho_o}{\delta_o|x|}\left(H_o-\frac{k_x}{2}x^2-\frac{k_y}{2}y^2-g_o\psi\right)\right]+1\right\}. \tag{21}$$

The two parameters f_o and H_o are related, respectively, to the density of the beam, and its size and emittance. They depend on each other through the normalization equation

$$N_L = \int\int n(x,y)dx\,dy$$
$$= \pi f_o \int\int \left[\mathrm{erf}\left(\frac{\lambda(x,y)\rho_o}{\delta_o|x|}\right)+1\right]dx\,dy, \qquad (22)$$

where N_L is the linear (longitudinal) density of the beam: It depends on the current I by the relation $N_L = I/(qv_z)$. When we solve Eq. (21) for different values of the parameter δ_o we will be interested in comparing solutions corresponding to beams that carry the same current. After setting f_o to a fixed value, we shall use Eq. (22) to determine H_o.

Finally, notice that in the limit $\delta_o \to 0$ Eq. (21) turns into Eq. (13), as expected, since

$$\lim_{\delta_o\to 0}\left[\mathrm{erf}\left(\frac{\tau}{\delta_o}\right)+1\right] = 2\mathscr{H}(\tau). \qquad (23)$$

A. Emittance calculation

The beam distribution can be characterized in terms of the emittance and related rms quantities. The rms emittance is defined by

$$\epsilon_x = (\langle x^2\rangle\langle p_x^2\rangle - \langle xp_x\rangle^2)^{1/2}$$

(analogous expression for ϵ_y).

For a KV round beam of radius a without dispersion [see Eq. (12) with $\delta_o = 0$], it is easily found that

$$\langle x^2\rangle_o = \frac{a^2}{4}, \quad \langle p_x^2\rangle_o = \frac{H_o}{2}, \quad \langle xp_x\rangle_o = 0.$$

Therefore, the emittance reads

$$\epsilon_{ox} = \frac{a}{2}\sqrt{\frac{H_o}{2}}. \qquad (24)$$

For the case with dispersion it is possible to reduce the expressions for the second moments to calculation of double integrals over x and y [11]:

$$\langle x^2\rangle = \frac{f_o\pi}{N_L}\int\int x^2\left[\mathrm{erf}\left(\frac{\lambda(x,y)\rho_o}{\delta_o|x|}\right)+1\right]dx\,dy, \qquad (25)$$

$$\langle p_x^2\rangle = \frac{f_o\pi}{N_L}\int\int \lambda(x,y)\left[\mathrm{erf}\left(\frac{\lambda(x,y)\rho_o}{\delta_o|x|}\right)+1\right]dx\,dy + \frac{f_o\sqrt{\pi}}{N_L}\int\int\frac{\delta_o|x|}{\rho_o}e^{-[\lambda(x,y)\rho_o/\delta_o|x|]^2}dx\,dy, \qquad (26)$$

$$\langle xp_x\rangle = 0. \qquad (27)$$

By the same token we can evaluate the rms value of the longitudinal momentum spread as a function of the parameter δ_o:

TABLE I. Parameters for the University of Maryland electron ring smooth model.

Beam energy E_o	10 keV
Tune ν_o	7.6
Focusing function k	17.437 m^{-2}
Radius of curvature ρ_o	1.82 m

$$\langle\delta^2\rangle = \frac{\delta_o^2}{2} - \frac{\delta_o^2}{2\sqrt{\pi}}\frac{f_o}{N_L}\int\int dx\,dy\left(\frac{\lambda(x,y)\rho_o}{\delta_o|x|}\right) \times e^{-[\lambda(x,y)\rho_o/\delta_o|x|]^2}. \qquad (28)$$

The normalization factor N_L is the same as in Eq. (22). In the limit of vanishing space charge $\psi\to 0$, the calculation of second moments can be carried out analytically. We found

$$\langle\delta^2\rangle = \frac{\delta_o^2}{2} + \frac{1}{4}\frac{\delta_o^4}{k\rho_o^2 H_o}, \qquad (29)$$

$$\langle x^2\rangle = \frac{a^2}{4} + \frac{\langle\delta^2\rangle}{k^2\rho_o^2} + \frac{1}{16}\frac{\delta_o^4}{k^3\rho_o^4 H_o}, \qquad (30)$$

$$\langle p_x^2\rangle = \frac{H_o}{2} + \frac{1}{16}\frac{\delta_o^4}{k^2\rho_o^4 H_o}, \qquad (31)$$

$$N_L = 2\pi^2 f_o\left(\frac{2H_o}{k} + \frac{\delta_o^2}{2k^2\rho_o^2}\right).$$

Notice the expressions above are consistent with the Eq. (7) we derived for a general distribution (i.e., in the expression for $\langle x^2\rangle$ the coefficient of $\langle\delta^2\rangle$ is D^2; the first correction to $\langle p_x^2\rangle$ is of order δ_o^4).

One of our goals will be to check the scaling of the rms quantities with respect to δ_o, when space charge effects are included.

VI. THE NUMERICAL SOLUTIONS

In solving numerically the nonlinear Poisson equation (21), we have used the successive overrelaxation method (SOR) described in [12] and recommended in [13].

The numerical solutions of the Poisson equation presented in this paper are based on a choice of parameters that mimic the design values of the Maryland Electron Ring [3] in the smooth approximation. See Table I. In the calculations we consider the case of a beam passing through a pipe of square cross section with side of length 2 cm. The wall of the pipe is assumed to be an equipotential boundary for the potential ψ. In this paper we show two sets of solutions.

The solutions of the first set have been obtained by varying the parameter δ_o, which is related to the rms value $\sigma_\delta \simeq \delta_o/\sqrt{2}$ of the longitudinal momentum distribution. In particular, δ_o ranges between 10^{-2} and 10^{-3}. The results, in terms of the horizontal profile $(y=0)$ of the beam density $n(x,y)$, see Eq. (20), are shown in Fig. 1.

The various solutions have been normalized in such a way that the corresponding beams carry the same current (I

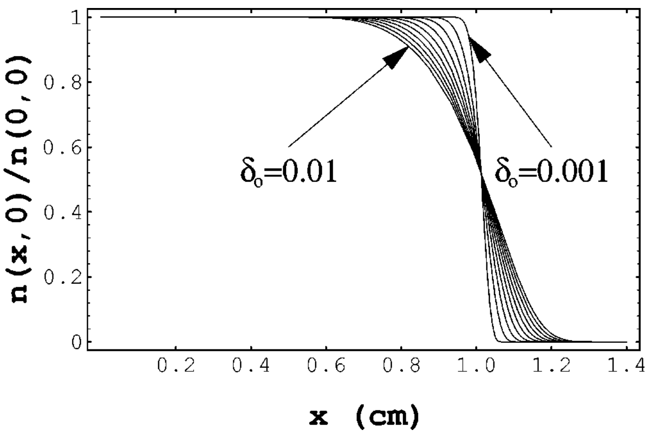

FIG. 1. Scaled density profile $n(x,y)/n(0,0)$ for the generalized KV beam at $y=0$. Ten density profiles are shown corresponding to values of δ_o ranging from 0.001 to 0.01 ($I=0.105$ A, $\nu/\nu_o=0.317$).

$=0.105$ A). In other words, the normalization constant N_L, see Eq. (22), is the same in all the cases. Since N_L depends on δ_o, in order to achieve the desired normalization the parameter H_o has to be properly tuned. The dependence of H_o on δ_o, however, turns out to be quite weak. The other parameter of the problem, f_o, has been kept fixed: f_o governs the peak value of the beam density and should be independent of δ_o.

The current we have chosen corresponds to a tune depression $\nu/\nu_o=0.317$. (The tune depression is evaluated for the KV beam in the limit $\delta_o \to 0$). Figure 2 shows the full density function in the (x,y) plane for $\delta_o=0.01$.

The rms values $\langle x^2 \rangle_{\delta_o}$ of the horizontal size of the beams and the horizontal emittances have been calculated according to Eq. (27) and the results shown in Fig. 3 and Fig. 4. In both pictures we plot the scaled values $x_{\rm rms}=(\langle x^2 \rangle_{\delta_o}/\langle x^2 \rangle_o)^{1/2}$ and ϵ_x/ϵ_{ox}, where $\langle x^2 \rangle_o$ and ϵ_{ox} are the values of the corresponding quantities for $\delta_o=0$ (standard KV beam). That is, the rms quantities are scaled with respect to the corresponding quantities of a KV beam in a straight channel under the same external focusing and with the same current.

The curves in the figures are the parabolic fitting obtained by retaining the first four points. As we can see, for small values of δ_o, $\delta_o < 0.005$, the rms quantities scale according to a quadratic power as in the limit of vanishing space charge [see Eqs. (30), (31)].

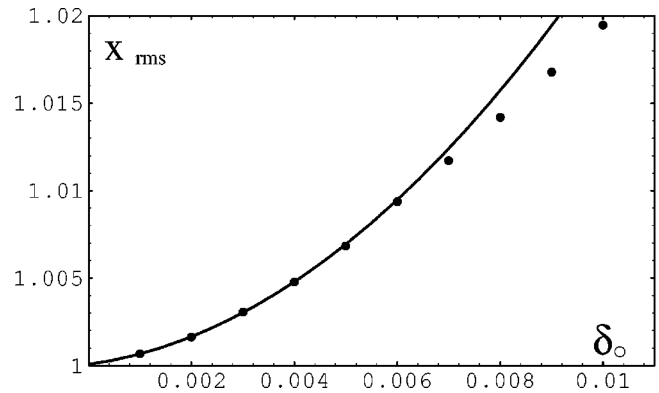

FIG. 3. Scaled rms horizontal size of the beam, $x_{\rm rms}=(\langle x^2 \rangle_{\delta_o}/\langle x^2 \rangle_o)^{1/2}$, as a function of δ_o ($I=0.105$ A, $\nu/\nu_o=0.317$).

In the second set of solutions the longitudinal momentum spread was kept fixed ($\delta_o=0.01$), while varying the value of the beam currents. Again, for a given choice of the current (or N_L) the normalization equation (22) has to be solved for H_o (the other parameter f_o is kept constant). The dependence of H_o on the current is related to the fact that as the space charge increases the emittance of the beam has to be decreased in order for the beam to maintain the same size. Indeed, Eq. (24) shows that in the absence of dispersion the emittance is proportional to the square root of H_o.

Ten different currents have been considered, ranging from $I=0.02$ to $I=0.112$ A, and corresponding to a tune depression ranging between $\nu/\nu_o=0.91$ and $\nu/\nu_o=0.20$. The values of the parameters for the various beams are summarized in Table II. The tune depression and emittance reported in the table refer to the corresponding KV beams in straight channels. The density profiles are shown in Fig. 5.

The rms values of the horizontal size of the beam have also been calculated and are shown in Fig. 6 [again, what we actually plot is the scaled quantity $x_{\rm rms}=(\langle x^2 \rangle_{\delta_o}/\langle x^2 \rangle_o)^{1/2}$]. The data shown in the picture allow one to check the adequacy of the first of Eqs. (7) to describe the horizontal rms size of the beam after the substitution of the undepressed

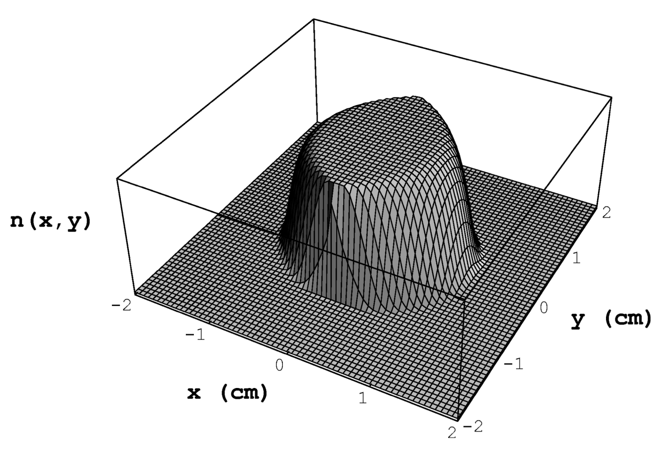

FIG. 2. Density distribution $n(x,y)$ for $I=0.105$ A, $\nu/\nu_o=0.317$, $\delta_o=0.01$.

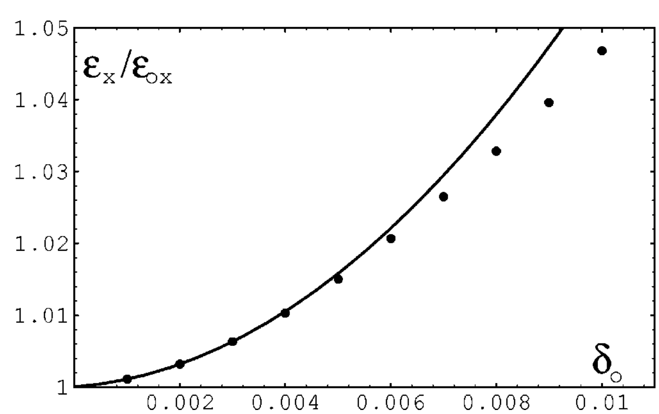

FIG. 4. Scaled horizontal rms emittance ϵ_x/ϵ_{ox}, as a function of δ_o ($I=0.105$ A, $\nu/\nu_o=0.317$).

TABLE II. Current, perveance, tune depression, and rms emittance for round KV beams of 10 keV electrons. Beam radius is 1 cm.

I (A)	K	ν/ν_o	ϵ_{ox} (mm mrad)
0.0200	0.2986×10^{-3}	0.910	95.03
0.0302	0.4513×10^{-3}	0.861	89.88
0.0404	0.6039×10^{-3}	0.809	84.40
0.0507	0.7565×10^{-3}	0.752	78.54
0.0609	0.9092×10^{-3}	0.692	72.22
0.0711	1.0618×10^{-3}	0.625	65.28
0.0813	1.2145×10^{-3}	0.551	57.51
0.0916	1.3671×10^{-3}	0.465	48.52
0.1018	1.5197×10^{-3}	0.358	37.41
0.1120	1.6724×10^{-3}	0.202	21.12

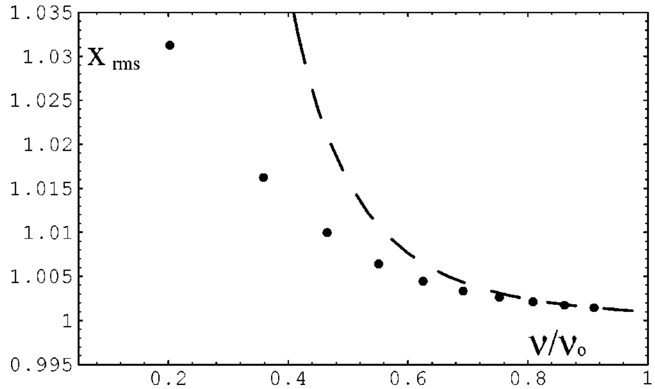

FIG. 6. Scaled horizontal rms size $x_{\mathrm{rms}}=(\langle x^2\rangle_{\delta_o}/\langle x^2\rangle_o)^{1/2}$ as a function of the tune depression ν/ν_o (dots). The dashed line represents Eq. (32).

tune with the depressed tune in the expression for the dispersion.

The dashed curve in the picture is

$$x_{\mathrm{rms}}=\left(1+\frac{\delta_o^2 D^2}{2\langle x^2\rangle_o}\right)^{1/2} \tag{32}$$

with $D=\rho_o/\nu^2$, where ν is the depressed tune evaluated for the KV beam in the absence of dispersion. Although Eq. (32) gives a good approximation if the tune depression is sufficiently high, for a smaller tune depression it provides a very large upper bound.

Finally the scaled emittance has also been calculated and plotted in Fig. 7.

VII. DISCUSSION

The main purpose of this study was to evaluate the scaling of the parameters characterizing the beam (rms size, emittance) as a function of the longitudinal momentum deviations and space charge. The stationary solutions of the Vlasov-Poisson equation have been calculated in the form of generalized KV beam distributions, which turn into the standard KV distributions in the limit of vanishing longitudinal momentum spread. For a given current we see that the presence of a finite momentum spread smooths the space distribution in the horizontal plane. As a result a tail appears, which is responsible for the rms growth of the horizontal plane. The tail profile has a rough exponential behavior, which is a reflection of the particular distribution of the momentum spread we chose. For small values of the rms momentum spread $\sigma_\delta\simeq\delta_o/\sqrt{2}$ the rms horizontal scales quadratically with respect to δ_o as expected.

The growth in the horizontal size of the beam, for a fixed momentum spread, is a function of the current. In particular, it increases with the current (i.e., it increases as the tune depression ν/ν_o decreases). This effect is expected. For larger currents the effective focusing on the particles due to the space charge is smaller and therefore particles off-momentum will tend to reach larger distances off-axis on the horizontal plane. However, as the beam spreads out transversely the charge density decreases and the net focusing gets less depressed. This mechanism explains the overestimate of the rms horizontal size growth, based on the simple replacement of the undepressed tune with the depressed tune in the expression for the dispersion function (Fig. 6, dashed line). For an attempt to give a more accurate analytical description of the rms quantities of the beam as a function of the tune depression, we refer to another paper [14].

Finally we want to remark that all the comparisons have been carried out between a solution of the full Vlasov-Poisson equation with the dispersive terms and a solution of

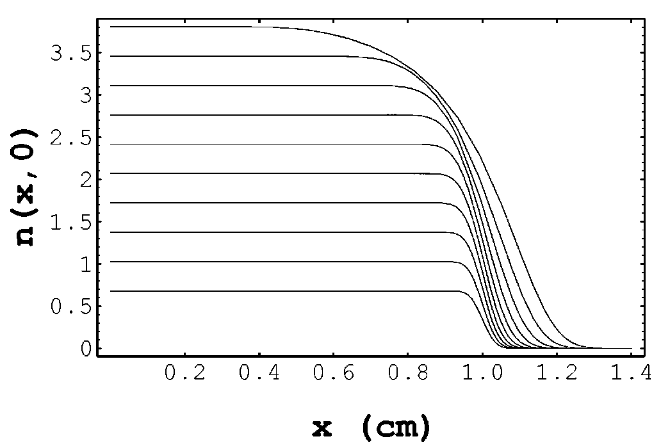

FIG. 5. Section of the density distributions $n(x,y)$ at $y=0$ for various beam currents (see Table II). The densities are in units of 10^7 particles/cm^3. $\delta_o=0.01$.

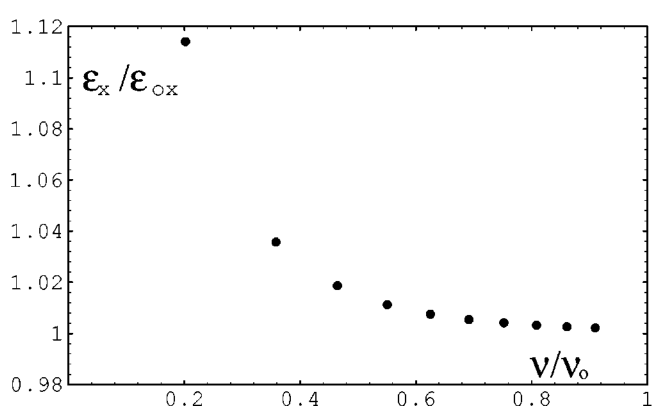

FIG. 7. Scaled horizontal rms emittance ϵ_x/ϵ_{ox}, as a function of the tune depression ν/ν_o.

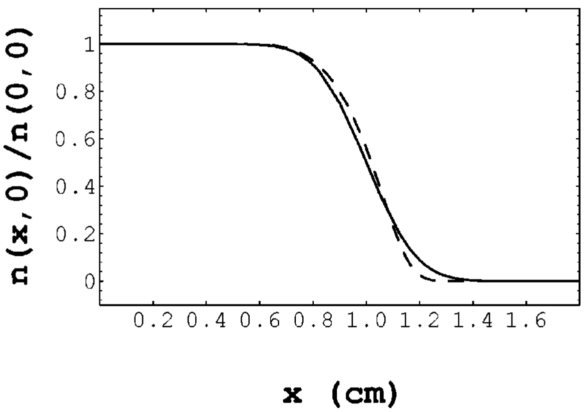

FIG. 8. Scaled density profile $n(x,0)/n(0,0)$ of the KV beam (simplified analytical model) after a matched injection (solid line) and profile corresponding to the stationary solution of the Vlasov-Poisson equation evaluated numerically (dashed line).

the same equation without a dispersive term (i.e., $1/\rho_o=0$) corresponding to a beam carrying the same current and exposed to the same external focusing. The last solution of course is just an ordinary KV beam. Both solutions are stationary solutions in smooth circular and straight channels, respectively.

It is interesting to question whether the generalized KV beam is the result of the evolution of the corresponding KV beam after injection into a dispersive channel, or more generally what is the relationship between the two. It is our conjecture that a least for relatively small currents the generalized KV beam should provide a close approximation of the stationary solution of the Vlasov equation that is achieved after a matched injection of an ordinary KV beam with Gaussian momentum spread into a circular channel. In the next section this issue is addressed by working out a simplified analytical model.

VIII. ''TIME'' DEPENDENT VLASOV EQUATION

So far we have been dealing with the problem of studying the stationary solutions of the Vlasov equation. However, as pointed out in the preceding section, it would also be interesting to investigate the evolution of a (standard) KV beam after injection into a smooth dispersive channel. The exact treatment of this problem would require the solution of the ''time'' dependent Vlasov equation (10) and related Maxwell equations. The problem would probably be best and most efficiently solved by using a particle in cell (PIC) code instead of directly trying to solve the required PDE.

In this section we work out a simplified analytical model and explore the two cases of matched and mismatched injection. In the model we assume that the only net effect of space charge is to depress the focusing function. Also, we neglect all the possible effects stemming from time-dependent fields.

First, let us introduce the symbol $\vec{\zeta}$ to denote the set of the dynamical variables. A certain distribution in the phase space at the initial location $z=z^o$, $g(\vec{\zeta}^o)$, evolves into a distribution $f(\vec{\zeta}^f,z^f)$ at $z=z^f$ given by

$$f(\vec{\zeta}^f,z^f)=g(\mathcal{M}^{-1}_{(z^o,z^f)}\vec{\zeta}^f), \tag{33}$$

where $\mathcal{M}_{(z^o,z^f)}$ is the transfer map that gives the position of any point in the phase space at $z=z^f$ as a function of the position in phase space at $z=z^o$:

$$\vec{\zeta}^f=\mathcal{M}_{(z^o,z^f)}\vec{\zeta}^o.$$

For a continuous bend of radius ρ, in the linear approximation the transfer map reads

$$x^f=x^o\cos\omega z^f+p_x^o\frac{1}{\omega}\sin\omega z^f+\frac{\delta}{\rho\omega^2}(1-\cos\omega z^f),$$

$$p_x^f=-x^o\omega\sin\omega z^f+p_x^o\cos\omega z^f+\frac{\delta}{\rho\omega}\sin\omega z^f,$$

$$y^f=y^o\cos\omega z^f+p_y^o\frac{1}{\omega}\sin\omega z^f,$$

$$p_y^f=-y^o\omega\sin\omega z^f+p_y^o\cos\omega z^f, \tag{34}$$

where ω is the depressed betatron frequency.

We consider two cases: a mismatched injection given by an abrupt transition from a straight injection line to the smooth circular channel and a matched injection. The matched injection can be obtained by a continuous bend with radius of curvature $\rho_B=2\rho_o$, the same focusing ω as in the circular machine and length $z_B=\pi/\omega=\lambda_b/2$ (λ_b is the betatron oscillation wavelength). In both cases we assume for the beam an initial KV distribution described by

$$g=\frac{f_o}{\delta_o\sqrt{\pi}}e^{-\delta^2/\delta_o^2}\hat{\delta}\left(\frac{1}{2}(p_x^2+p_y^2)+\frac{\omega^2}{2}(x^2+y^2)-H_o\right). \tag{35}$$

Next we want to calculate the evolution of the beam density after injection. Consider the matched case first. After evaluating the distribution function at $z=z^f<z_B$ using Eqs. (33) and (35) and the expression for the transfer map (34), we integrate over p_x,p_y,δ to find the corresponding space density to be

$$n(x,y)=f_o\pi[\mathrm{erf}(\tau_+(x,y,z))-\mathrm{erf}(\tau_-(x,y,z))]. \tag{36}$$

Here τ_+ and τ_- are defined by

$$\tau_\pm(x,y,z)=\omega^2\frac{\rho_B}{\delta_o}\frac{x\pm\sqrt{a^2-y^2}}{(1-\cos\omega z)}, \tag{37}$$

with a being the radius of the KV beam at injection. At $z=z_B$ we get a profile that matches the stationary beam density within the circular channel.

Comparison (Fig. 8) with the density profile obtained by numerical solution of the Vlasov-Poisson equation will allow us to determine $k_{\mathrm{eff}}=\omega^2$, by numerical fitting. For the case $\delta_o=0.01$ and $I=1.05$ A, we find that we obtain relatively similar profiles if $k_{\mathrm{eff}}=2.63\ \mathrm{m}^{-2}$. If k_{eff} were calculated using an ideal KV beam model with radius a using Eq. (8), we

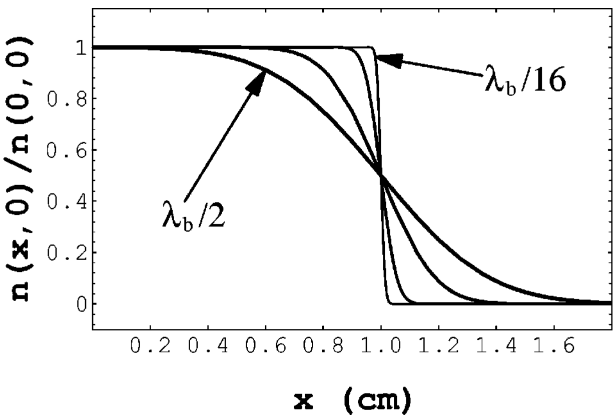

FIG. 9. Evolution of a KV beam with Gaussian longitudinal momentum spread injected into a smooth dispersive channel (simplified analytical model). Mismatched injection. The four beam density profiles correspond to $z=\lambda_b/16,\lambda_b/8,\lambda_b/4,\lambda_b/2$, where λ_b is the betatron oscillation wavelength.

would find the result $k_{\mathrm{eff}}=1.75\ \mathrm{m}^{-2}$. The deviation between the two values can be justified in terms of the enlargement of the horizontal cross section of the beam and consequent decrease of the space charge forces. In other terms, as we have already noticed, the effective tune, because of dispersion, is larger than before injection.

In the case of a mismatched injection the density is defined by Eqs. (36) and (37) with ρ_o replacing ρ_B. The formula shows that the beam pulses around the equilibrium profile corresponding to the matched beam.

The profiles at four different values of z $(\lambda_b/16,\lambda_b/8,\lambda_b/4,\lambda_b/2)$ are shown in Fig. 9. At $z=\lambda_b/4$ the density profile is the same as in the case of a matched injection (solid line in Fig. 8); at $z=\lambda_b/2$ it reaches its maximum extension.

IX. CONCLUSIONS

In this paper we have shown and discussed stationary solutions of the Vlasov-Poisson equation describing beams of charged particles in the presence of dispersion. In particular, we have considered solutions that provide the generalization of the KV beam. The numerical solutions presented in the paper show, as expected, that dispersion reshapes and enlarges the beam distribution. The growth, however, evaluated with respect to ordinary KV beams obtained as solution of the same Vlasov equation with vanishing dispersive term, is modest. In a strong focusing lattice $(\nu_o \gg 1)$, for a tune depression (ν/ν_o) in the range 0.2–0.4 the growth for emittance and rms horizontal radius is of the order of or below 10%. One important lesson learned from the study is that for highly space charge dominated beams, the dispersion function cannot be calculated simply by changing the undepressed tune with depressed tunes in the formulas. This result, obtained here in the smooth approximation, has motivated us to develop a method to calculate (approximately) the dispersion function in the more general case with a z dependence, which is relevant for matching purposes [14].

ACKNOWLEDGMENTS

We thank Nathan Brown for suggesting the use of the SOR method and J. G. Wang for useful discussions. The work was supported by the U.S. Department of Energy.

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[7] This is the case, for example, for the generalized KV beam distribution introduced in Sec. V in the limit of vanishing space charge.

[8] A KV beam has the property that forces due to space charge are linear. The distribution is defined in Eq. (12) with $\delta_o=0$. See also [4,6].

[9] If in the absence of dispersion the beam has a KV distribution, the depressed tune is a well defined quantity since all the forces are linear. If it does not have a KV distribution, the depressed tune should be interpreted in a rms sense, using the concept of equivalent KV beam [6], Chap. 5.

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